# Why you’re less popular than your friends - and through no fault of your own

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Think about your closest friends. Now think about how many friends they have. Now think about how many *they* have. The chances are, your friend, and then your friend’s friend, have more friends than you – it’s rough, isn’t it? More people like them, more people socialise with them, and more people spend time with them.

Despite seeming rather depressing, this has actually been noted in not just close friendships, but all situations like it – with the topology of most networks showing that your connections probably have more connections to others than you do. But don’t take it personally – it’s mathematics that forces this outcome, no matter who’s chosen and who is friends with whom.

This is the friendship paradox, first proven in 1991 by sociologist Scott Feld, which seems impossible and illogical until it all suddenly clicks and makes sense. In this essay, I intend to introduce you to basic graph theory, how this presents itself in real world scenarios, and, ultimately, make the paradox click for you and show you that it isn’t a social failing, but you, like all of us, have simply fallen victim to mathematics and inevitability[1].

## *A brief introduction to graphs*

Before delving deeper into technicalities, we first need to have a mutual understanding of terminology and practical examples of it. Throughout the rest of the essay, I will refer to this graph, which is an extremely basic example of a group of friends and their connections.

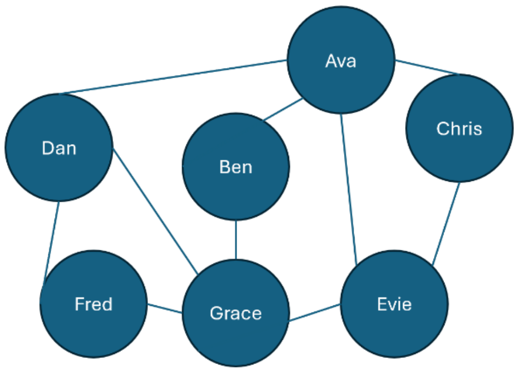


*Figure 1*

[1] I intend to do this in the simplest way possible, keeping the essay accessible to anyone who wants to read it, no matter their prior maths knowledge, whilst maintaining the ‘mathsy’ element behind it!

So, what is a graph? It is a structure comprised of nodes (points) and edges (their connections) to other nodes used to model overall relationships. This model doesn't need to scale, nor does it need to be accurately positioned relative to the other nodes, it just needs to provide a representation – think of it as a topological map like the London tube map. In the example above, the nodes are Ava, Ben, Chris, Dan, Evie, Fred and Grace, with the edges being the lines between the nodes, representing who every person is friends with. Every node also has a value associated with it, called its degree/valency – this just means how many connections it has with other neighbouring nodes. For example, Ava has a valency of 4, whereas poor old Ben only has a valency of 2.

## *Making the paradox clear*

With the key terminology now defined, I can go about explaining the paradox in a more complete way and reveal the logic behind it. The paradox states that for every individual node, it will statistically have a smaller average valency than most of the other nodes that it is directly connected to via a single edge – meaning in the friendship paradox that if you randomly take one of your friends, they likely have more friends than you. Looking back at *Figure 1* and writing out a table, we can see the following:

| Node | Valency | Average Valency of Friends | Friends Difference |
|---|---|---|---|
| Ava | 4 | 2.5 | +1.5 |
| Ben | 2 | 4 | -2 |
| Chris | 2 | 3.5 | -1.5 |
| Dan | 3 | 3.3 | -0.3 |
| Evie | 3 | 3.3 | -0.3 |
| Fred | 2 | 3.5 | -1.5 |
| Grace | 4 | 2.5 | +1.5 |

*Figure 2*

**In the table the average valency of friends is calculated by adding the valency of all friends of the individual and then dividing by the individual's valency (meaning the number of friends they have)*

From this we can work out the average valency of an individual and the overall average valency of friends:

## Calculating average valency of an individual

To calculate the average, or mean ($\bar{x}$), valency of an individual, we divide the sum ($\sum$) of all the degrees of the individuals (where $d_i$ represents degree of an individual) and then divide by the total number of individuals (n). This gives us:

$$\bar{x}_{d(i)} = \frac{\sum d_i}{n}$$

$$\bar{x}_{d(i)} = \frac{4+2+2+3+3+2+4}{7}$$

$$\bar{x}_{d(i)} = 2.9$$

## Calculating overall average valency of friends

Unlike calculating the mean valency of an individual, to calculate the overall mean valency of friends ($\bar{x}_{vof}$) we can't just add the average valency of friends and then divide by the total, 7. That would just lead to us having an average of a bunch of averages, rather than an *overall* average for all the friends of an individual.

To calculate this, we must sum all the valencies, of all the neighbouring nodes to each individual node – this may not be too challenging for our simple graph in *Figure 1,* but it can become much longer if it's a larger, more realistic graph.

Luckily, there is a shortcut: what we are really doing is adding the sum of the valency of any individual multiplied by the number of times that individual appears, as its degree will need to be added on for every time it is viewed as a valency of a friend of a different node. For example, Ava has a valency of 4 so her degree will be counted once for each of her 4 friends, making it 4 X 4. This can be simplified to:

$$(d_i)(d_i) = (d_i)^2$$

Once we have the sum of valencies squared, we must divide that by the total valency across the graph as a whole ($\sum d_i$) – note how this is just double the total number of edges in the graph, as every edge must be counted twice for both directions that the friendship 'travels' in – this is otherwise known as Euler's Handshaking Lemma. In other words, we are working out the total degree of all the friends of all the individuals in the graph and dividing that by all the friendships between everyone.

This therefore makes the final formula:

$$\bar{x}_{vof} = \frac{\sum(d_i)^2}{\sum d_i}$$

$$\bar{x}_{vof} = \frac{62}{20}$$

$$\bar{x}_{vof} = 3.1$$

From looking at the final column in *Figure 2* alone, you can see that for almost everyone there is a negative friends difference, meaning people tend to have fewer friends than their friends do. After calculating the averages, you can see that despite everyone being connected and all on the same graph, the average valency of an individual is only 2.9, whereas the average valency of any individual's friends' is 3.1. Though this difference may seem small, it's clear that you will statistically have less friends on average than your friends do – and the larger the network gets, the larger the gap slowly grows.

## *How does this always happen and what's the reason why – in real words?*

So, what's the proof and why does this always happen? You first need to know the formula for variance ($\sigma^2$) – variance just meaning how spread out the valency of each node is from the mean. The formula is:

$$\sigma^2 = \frac{\sum(d_i)^2}{n} - \bar{x}_{d(i)}{}^2$$

Re-arranging for the sum of all the valencies of all the neighbouring nodes to each individual node gives us:

$$\sum(d_i)^2 = (n)(\bar{x}_{d(i)}{}^2 + \sigma^2)$$

We can substitute that equation into the equation we want to find for the overall average valency of friends. This will give us:

$$\frac{\sum(d_i)^2}{\sum d_i} = \frac{(n)(\bar{x}_{d(i)}{}^2 + \sigma^2)}{(n)(\bar{x}_{d(i)})}$$

$$= \bar{x}_{d(i)} + \frac{\sigma^2}{\bar{x}_{d(i)}}$$

And that's it! We just proved that the overall average valency of friends is the mean valency *plus* another term – and this other term will always be positive given it's the variance divided by the mean valency, both of which are positive, therefore a positive value – meaning that for every graph[2], the adjacent nodes will statistically have a higher degree than the chosen node on average.

You may now have the maths, but what about the logic that will help you wrap your head around it – here's the secret.

Let's look at Ava and Grace in our example, they're the joint most popular friends – the nodes with the highest degrees – thus they appear in many peoples list of friends. That means if you take a random node, there is a high chance they are connected to them via a single edge therefore increasing the average valency of friends of any chosen node. As a result, over the total graph, Ava and Grace are skewing the average by appearing in many peoples list of friends whilst someone with only a couple of friends – Ben for example – is barely accounted for in the average as they are only included twice rather than being repeated multiple times like Ava and Grace. This is exactly what the variance term in our proof represents – the more unbalanced a graph and network is, the more the high degree, popular nodes dominate and the larger the gap becomes between your perceived popularity and theirs.

## *The friendship paradox applied – and how it can save lives*

There are many applications for the friendship paradox, from network security and marketing to polling bias and financial networks, and without realising, the paradox affects our everyday lives. However, there is one application that I think is much more interesting and relevant in a post-covid society – how the friendship paradox can be used for the most efficient vaccination.

Imagine a situation where vaccinations are scarce and you need to distribute them strategically (indeed no imagination needed given the recent meningitis scare in Kent). One's gut reaction might be to provide randomly on a first come first serve basis or to those who are more 'at risk'. This, however, won't be the optimal method as there is an equal chance the person you randomly select has a high valency or a low valency – you could just be vaccinating people who wouldn't be massive spreaders of the disease in the first place.

In this case, the friendship paradox offers a simple and elegant solution that can do all the hard work. Rather than vaccinating random individuals, vaccinate one of their friends. Again, this appears counter intuitive but is mathematically correct. As we have already proven, this will result in us vaccinating individuals with a higher degree – therefore automatically targeting those who are more likely to be spreaders of the

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[2] Other than in a complete graph – a graph where every node is connected directly via one edge to every other node – as this would make the average valency *n-1* and the overall average valency of friends also *n-1*.

disease without having to map out the whole network. While this may not always result in vaccinating *exactly* all the most connected individuals, the strategy makes it straightforward to approximate the optimal result, even in large and complicated networks.

Not only does vaccinating the most connected individuals decrease the risk of their spreading it to others, but it also reduces the risk of other higher connected individuals catching it and spreading it further – in turn massively reducing the rate at which it spreads through one tactical vaccination.

To me, the most remarkable part is that you don't need any prior knowledge of the network to exploit it. No data collection, no large-scale mapping, no identifying superspreaders – just the friendship paradox doing its thing and saving lives in the process.

## *To wrap it all up*

This essay started as an almost whimsical observation of how your friends always seem *just that little bit* more popular than you are. But this question about social networks goes much deeper: the structure of any network and graph, whether it's the number of friends you have or spread of disease during a pandemic, is quietly ruled by the same elegant maths.

At its heart, the friendship paradox is a story about variance, inequality and how that ultimately guarantees those popular nodes or individuals dominate everyone else's experience and – as we have seen – this quirk might be positively exploited to have a real-world impact and help save lives.

Graph theory may initially seem like an abstract corner of mathematics that isn't particularly relevant to those of us who aren't data scientists, but, with Avas, Bens and Graces being everywhere, this area of maths is both beautiful and vital for our everyday lives.

So, the next time you feel like everyone around you is more popular than you – you're probably right. *However,* you now know why and, perhaps more remarkably, how this could save lives – and consequently you may even get a few more friends as a result!