Soap Films That Quit Easy

A short tour of the calculus of variations, told through the problem of finding the minimal surface between two parallel rings

Dip two parallel wire rings into soapy water, pull them slowly apart, and a soap film will stretch itself between them, forming a graceful curved waist that is narrower in the middle than at the ends. Keep pulling, and at some precise moment the film pops and collapses into two disconnected flat disks, one clinging to each ring. Both of these behaviours, the shape of the film while it lasts and the moment at which it gives up, can be predicted from first principles using a branch of mathematics called the calculus of variations, and the prediction can be checked numerically in about thirty lines of Python PyCharm. This is convenient, because the alternative involves me, a kitchen sink, and a degree of manual dexterity I do not possess.

A very short introduction to the calculus of variations

Ordinary calculus is about finding the number that makes a function as large or as small as possible. We pick a function *f(x)*, compute *f'(x)* - the derivative of *f(x)*, set it to zero, and read off the answer (the derivative, sometimes also written as *dy/dx* is the rate at which a function *f(x)* or *y* changes as *x* changes). The **Calculus of Variations** asks a slightly more ambitious question. Instead of searching for a number that extremises a function, we search for a whole function that extremises something called a functional.

A **functional** is a rule that takes a function as input and returns a single number. The most familiar form is an integral,

$$\textbf{(0)}\quad J[y] = \int_a^b L(x,\, y,\, y')\ dx$$

(all the curly bracket means is you **integrate** *L*. Integration just means finding the area under a graph: in this case, *L* between *a* and *b*. It is the opposite of finding the derivative)

where the integrand *L*, called the **Lagrangian**, depends on the unknown function *y*, its derivative *y'*, and possibly the independent variable *x* itself. The question is: of all the functions *y(x)* satisfying whichever boundary conditions we impose, **which one makes *J[y]* smallest** (or largest, or stationary)?

The strategy, due to Euler in the 1740s, is the natural generalisation of ordinary calculus. In one-variable calculus we look for points where nudging *x* leaves *f* unchanged i.e. where *dy/dx=0*. In the calculus of variations we look for functions *y(x)* where wobbling *y* leaves *J* unchanged. After some careful bookkeeping, the condition this imposes on y turns out to be a single differential equation (an equation which relates a function to its derivative), the famous **Euler-Lagrange equation:**

$$\textbf{(EL)}\quad \frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0$$

(while the curly *dy/dx* looks confusing, it just means the derivative of L when only what's at the bottom changes)

Any y(x) that extremises J must satisfy this. Despite its modest appearance, it is the workhorse of classical mechanics, optics, and general relativity, each of which supplies its own Lagrangian and bolts it into the same machinery.

One shortcut will save work in a moment. When L happens not to depend on x explicitly (only on y and y′), the Euler-Lagrange equation can be integrated once to give the **Beltrami identity:**

$$\textbf{(B)}\quad L - y'\left(\frac{\partial L}{\partial y'}\right) = C$$

This converts a second-order (involving the 2nd derviative) differential equation into a first-order (involving only the 1st derivative, which is generally much easier to solve. Armed with the Euler-Lagrange equation, the Beltrami identity, and a reasonable integrand, the general procedure for a variational problem is straightforward. Write down the quantity to be minimised as an integral, identify the Lagrangian, apply one of the two equations above, and solve the resulting ODE. In the case of the soap film, the procedure runs through cleanly and produces a classical result in about five lines.

Setting up the soap film

A soap film strung between two rigid boundaries will settle into whichever shape minimises its total surface area, because surface tension does a good job of contracting the film at every instant. Assume the setup is rotationally symmetric: two rings of equal radius R, parallel to each other, separated by a distance $2h$, with their common axis taken to be the *z-axis*. Let $r(z)$ denote the radius of the film at height z. The film is then a surface of revolution of the curve $r(z)$ about the *z-axis*, and the standard formula from A-level calculus gives its area as

$$\textbf{(1)}\quad A[r] = 2\pi \int_{-h}^{h} r(z)\sqrt{1 + r'(z)^2}\, dz$$

subject to the boundary conditions $r(-h) = r(h) = R$. This is exactly the sort of functional Euler and Lagrange had in mind. The Lagrangian is $L(r, r') = r\sqrt{(1 + r'^2)}$, which has no explicit dependence on z, so the Beltrami identity applies immediately and collapses the problem to the equation

$$\textbf{(2)}\quad \frac{r}{\sqrt{1+r'(z)^2}} = a$$

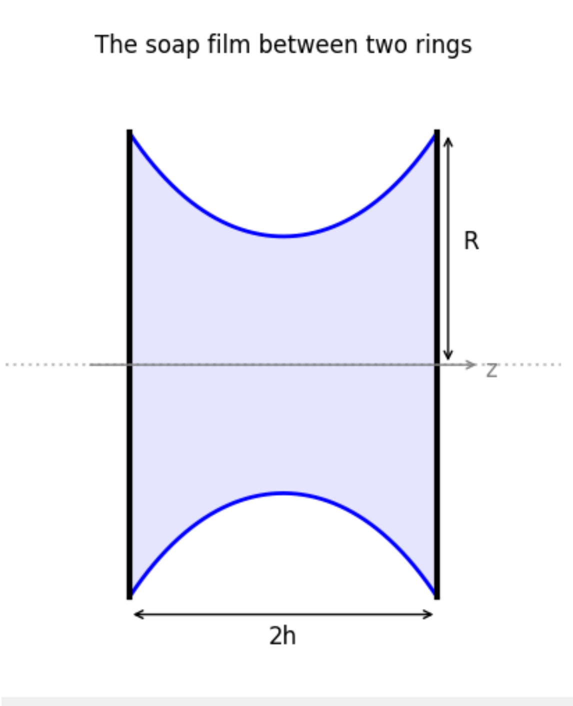
where a is a constant of integration fixed by the boundary conditions. Rearranging for r', separating variables and integrating produces the closed-form solution

$$\textbf{(3)}\quad r(z) = a\cosh\left(\frac{z}{a}\right)$$

which is known as the **catenoid**. It is the only smooth surface of revolution whose mean curvature vanishes everywhere, and it is therefore the unique family of candidate shapes our soap film is allowed to choose from.

[FIGURE 1. The figures following will have the output and the code side by side.]

The plot draws the two rings, the catenoid surface between them using the formulas we derived. A photograph of a real soap film would be more impressive, but producing one requires a steady hand, which is a supply I do not have in stock.




```
import numpy as np
import matplotlib.pyplot as plt
a = 0.6
h = 0.72 #                    Hii :)))
z = np.linspace(-h, h, num: 200)#     \(.> ᴗ < )/
r = a * np.cosh(z / a)
R = a * np.cosh(h / a)
fig, ax = plt.subplots(figsize=(6, 5))
ax.plot( *args: z, r, 'b-', linewidth=2)
ax.plot( *args: z, -r, 'b-', linewidth=2)
ax.fill_between(z, -r, r, alpha=0.1, color='blue')
ax.plot( *args: [-h, -h], [-R, R], 'k-', linewidth=3)
ax.plot( *args: [h, h], [-R, R], 'k-', linewidth=3)
ax.axhline( y: 0, color='gray', linestyle=':', alpha=0.5)
ax.annotate( text: '', xy=(h + 0.2, 0), xytext=(-h - 0.2, 0),
            arrowprops=dict(arrowstyle='->', color='gray'))
ax.text(h + 0.22, -0.06, s: 'z', fontsize=12, color='gray')
ax.annotate( text: '', xy=(-h, -R - 0.08), xytext=(h, -R - 0.08),
            arrowprops=dict(arrowstyle='<->'))
ax.text( x: 0, -R - 0.22, s: '2h', ha='center', fontsize=12)
ax.annotate( text: '', xy=(h + 0.05, 0), xytext=(h + 0.05, R),
            arrowprops=dict(arrowstyle='<->'))
ax.text(h + 0.12, R / 2, s: 'R', fontsize=12)
ax.set_xlim(-1.3, right: 1.3)
ax.set_ylim(-1.4, top: 1.4)
ax.set_aspect('equal')
ax.axis('off')
ax.set_title('The soap film between two rings')
plt.tight_layout()
plt.show()
```

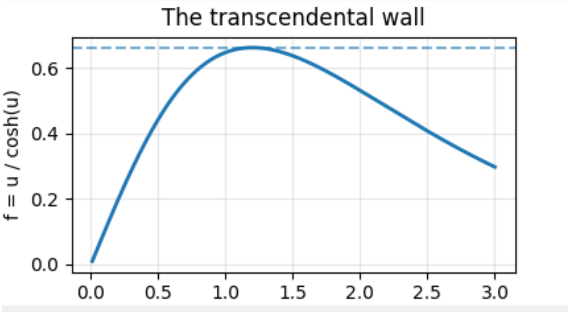
The transcendental wall

The catenoid formula in **(3)** still contains the free constant a, which has to be chosen so that the curve actually passes through the two rings at $z = \pm h$. Imposing $r(\pm h) = R$ gives

$a\cosh(h/a) = R$. Introducing the dimensionless variable $u = h/a$ tidies this up into

$$\textbf{(4)}\quad \frac{h}{R} = \frac{u}{\cosh(u)}$$

Given h and R, we want to solve this equation for u. There is no closed form, because the relation is transcendental - which is a way of saying that no amount of algebra will turn it into a formula for u in terms of h/R. But the behaviour of the right-hand side is easy to see graphically.




```python
import numpy as np
import matplotlib.pyplot as plt
u = np.linspace(0.01, 3, 500)
f = u / np.cosh(u)#             I wonder if anyone will
plt.figure(figsize = (6,4))#    even read the code (╥—╥)
plt.plot(u, f, linewidth = 2)
plt.axhline(f.max(), linestyle = '--', alpha = 0.6)
plt.xlabel('u = h / a')
plt.ylabel('f = u / cosh(u)')
plt.title('The transcendental wall')
plt.grid(alpha = 0.3)
plt.show()
```

[FIGURE 2.]

The plot reveals something interesting. For any value of *h/R* below the peak, the horizontal line $y = h/R$ intersects the curve at two distinct values of u, which correspond to two different catenoids: a fat, stable one and a thin, unstable one. For *h/R* above the peak, the horizontal line misses the curve altogether, and the transcendental equation has no solution. No catenoid of any shape can span the two rings.

Locating the peak exactly

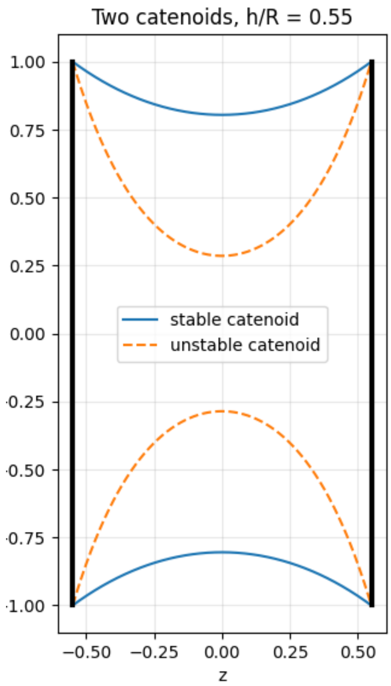
To pin down the peak, set the derivative of $f(u)$ equal to zero. A short calculation yields the condition $u = coth(u)$, which is again transcendental and again has no closed form. Newton's method handles it cheaply: start with a reasonable guess for u, compute how far the guess is from satisfying the equation (call this g), and how sensitive that error is to small changes in u (call this g'). The update $u \rightarrow u - \frac{g}{g'}$ produces a noticeably better guess. Repeat until the number stops moving.

```python
import numpy as np
'''
Solve u = coth(u) by Newton's method
on g(u) = u - coth(u)
Another common Newton W (ง -ᴗ- )✧
'''
u = 1.5  # initial guess
for _ in range(20):
    g = u - 1.0 / np.tanh(u)
    gp = 1.0 + 1.0 / np.sinh(u) ** 2
    u -= g / gp
u_star = u
ratio = u_star / np.cosh(u_star)
print(f'u*           = {u_star:.10f}')
print(f'(h/R)_crit   = {ratio:.10f}')
print(f'textbook val = 0.6627434193')
```

Running the script gives $u^* \approx 1.1996786$ and (h/R) *critical* ≈ 0.6627434, matching the published value to every digit printed. In plain terms, the calculus of variations, starting from nothing more than the single instruction to minimise the surface area integral, has told us that two rings separated by more than about 1.325 ring-radii can no longer support a catenoid soap film at all. This is a clean, quantitative, physically testable prediction obtained from a one-line Lagrangian.

Visualising the two catenoids

A check is to draw the two catenoid solutions side by side for a sub-critical value of h/R, to see that the same boundary condition really does admit two distinct profiles. Below the peak, the equation u / cosh(u) = h/R has a smaller root (the fat, stable catenoid) coloured blue and a larger root (the thin, unstable one) coloured orange, and we can plot both.



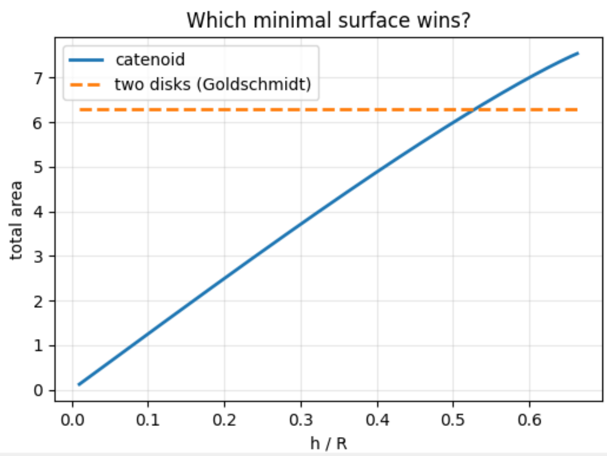

```python
import matplotlib.pyplot as plt
R = 1.0
target = 0.55  # h/R, sub-critical
# Find the two roots of u/cosh(u) = target by bisection
def f(u): return u / np.cosh(u) - target
# Stable (small u) root: bracket [0.01, 1.1996]
a1, b1 = 0.01, 1.1996
for _ in range(80):
    m = 0.5 * (a1 + b1)
    if f(a1) * f(m) < 0:
        b1 = m
    else:
        a1 = m
u_stable = 0.5 * (a1 + b1)
# Unstable (large u) root: bracket [1.1996, 5]
a2, b2 = 1.1996, 5.0
for _ in range(80):
    m = 0.5 * (a2 + b2)
    if f(a2) * f(m) < 0:
        b2 = m#                    I swear this is genuinely easier
    else:     #                    than blowing bubbles (,,>__<,,)
        a2 = m
u_unstable = 0.5 * (a2 + b2)
a_s = R / np.cosh(u_stable)
a_u = R / np.cosh(u_unstable)
h = target * R
z = np.linspace(-h, h, 400)
plt.figure(figsize=(6, 5))
plt.plot(z, a_s * np.cosh(z / a_s), label='stable catenoid')
plt.plot(z, -a_s * np.cosh(z / a_s), color='C0')
plt.plot(z, a_u * np.cosh(z / a_u), '--', label='unstable catenoid')
plt.plot(z, -a_u * np.cosh(z / a_u), '--', color='C1')
plt.plot([-h, -h], [-R, R], 'k-', linewidth=3)
plt.plot([h, h], [-R, R], 'k-', linewidth=3)
plt.gca().set_aspect('equal')
plt.legend()
plt.title(f'Two catenoids, h/R = {target}')
plt.xlabel('z')
plt.ylabel('r')
plt.grid(alpha=0.3)
plt.show()
```

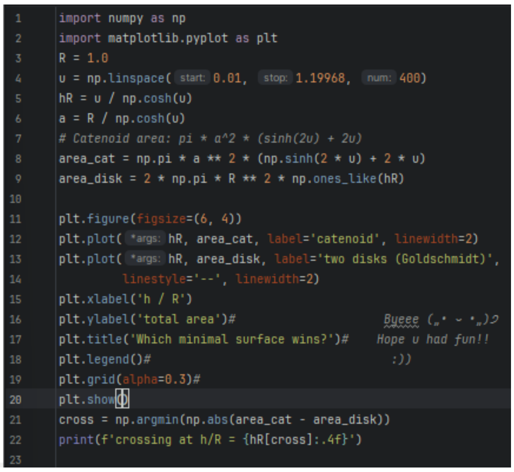
[FIGURE 3.]

The Goldschmidt get-out clause

The variational problem, as posed, tacitly assumes the soap film is a connected surface of revolution. Nothing in the physics actually enforces this. The film has a second configuration available to it from the very beginning: it can collapse into two disjoint flat disks, one inside each ring. Each disk is itself a minimal surface (both have zero mean curvature), and together they form a perfectly valid candidate minimiser. The total area of this configuration is simply $2\pi R^2$, independent of h.

This configuration, first noted by Carl Wolfgang Benjamin Goldschmidt in 1831, changes the story. There are now two competing candidates: the catenoid, whose area depends on h, and the two-disk Goldschmidt solution, whose area does not. Whichever has the smaller area is the configuration the soap film will, in principle, prefer.





```python
import numpy as np
import matplotlib.pyplot as plt
R = 1.0
u = np.linspace( start: 0.01,  stop: 1.19968,  num: 400)
hR = u / np.cosh(u)
a = R / np.cosh(u)
# Catenoid area: pi * a^2 * (sinh(2u) + 2u)
area_cat = np.pi * a ** 2 * (np.sinh(2 * u) + 2 * u)
area_disk = 2 * np.pi * R ** 2 * np.ones_like(hR)

plt.figure(figsize=(6, 4))
plt.plot( *args: hR, area_cat, label='catenoid', linewidth=2)
plt.plot( *args: hR, area_disk, label='two disks (Goldschmidt)',
         linestyle='--', linewidth=2)
plt.xlabel('h / R')
plt.ylabel('total area')#                      Byeee („• ᴗ •„)੭
plt.title('Which minimal surface wins?')#    Hope u had fun!!
plt.legend()#                                    :))
plt.grid(alpha=0.3)#
plt.show()
cross = np.argmin(np.abs(area_cat - area_disk))
print(f'crossing at h/R = {hR[cross]:.4f}')
```

[FIGURE 4.]: The script walks u along the stable branch of the catenoid, computes the corresponding h/R at each point, evaluates the catenoid's total area using the standard integral result π a² (sinh(2u) + 2u), and plots it against the flat line 2πR² representing the two-disk Goldschmidt solution. The crossing of the two curves is the interesting point, and the script prints its location.

The two curves cross at $h/R \approx 0.5284$, considerably before the catenoid ceases to exist at $h/R \approx 0.6627$. In the window between these two values, the catenoid still exists and is still locally stable, but the Goldschmidt two-disk configuration has strictly smaller total area. In that regime the catenoid is metastable, in the same sense that a pencil balanced on its point is metastable: left undisturbed it survives, but any perturbation will tip it into the lower-area configuration.

This explains a quiet discrepancy between the idealised prediction and the behaviour of actual soap films. In practice, a real film rarely survives all the way to $h/R = 0.6627$ before snapping. Any draught, vibration, or slight irregularity of the rings supplies the perturbation required to push the metastable catenoid over the edge, and the film rearranges itself into the Goldschmidt configuration somewhere inside the window between 0.5284 and 0.6627. The popping noise that accompanies the snap is, in a reasonably literal sense, the sound of a functional finding a better minimum.

Closing remarks

In the space of a few pages, the calculus of variations has produced three quantitative predictions about a soap film. It identifies the family of shapes the film is allowed to take (catenoids). It locates the ratio $h/R \approx 0.6627$ at which no such shape can span the two rings.

And it identifies a second, smaller ratio $h/R \approx 0.5284$ at which a disconnected configuration first begins to beat the catenoid on total area. All three results follow from a single integral and two classical ODE manipulations whose names have been attached to them for more than two centuries.

The only input supplied to the mathematics was the instruction to minimise surface area. No physical law about soap, water or surface tension was invoked at any stage. What comes out the other end is a complete description of how a real mechanical object will behave, and the agreement between the prediction and a Python script is exact to the precision of the arithmetic. The same general procedure, applied to different Lagrangians, governs light rays in optics, paths of least action in mechanics, and the shortest routes through curved spacetime in general relativity. The soap film is simply the cheapest laboratory in which to see it work. Because, while soap films have a tendency to quit easy, Mathematics never gives up.

References

Euler, L. (1744). Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes. Lausanne and Geneva.

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