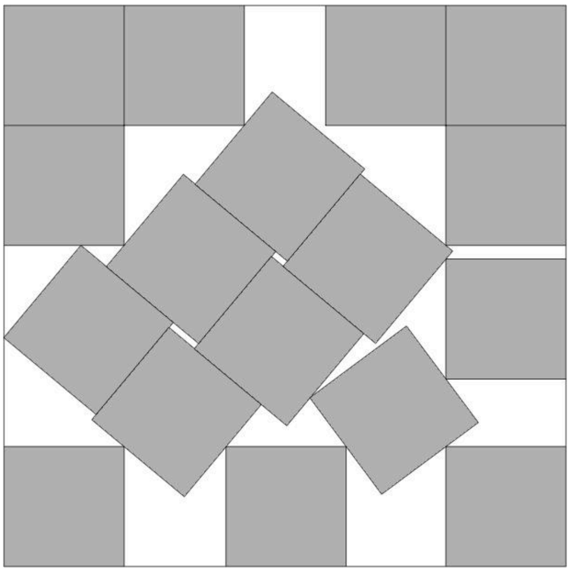
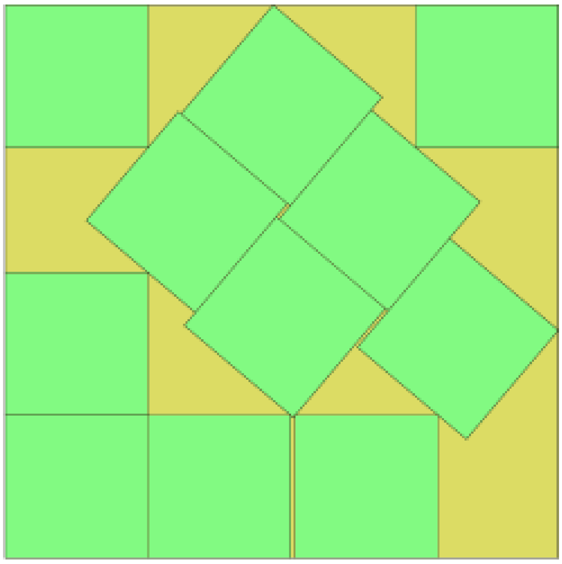
How many Squares would a Square-packer pack if a Square-packer could pack Squares?

1. Introduction

If you have spent any length of time on the internet in recent years, you have likely encountered a mention or two of the ‘square-packing’ problem. Many social media users lament that the solutions for certain square-packing problems are ‘deeply unsettling’ thanks to their asymmetric nature.

The intention of the square-packing problem is to fit *n* unit squares within a larger square of minimum side length. Pictured below, for reference, is the optimal packing for 17 unit squares, and 11 unit squares, in their respective minimum areas.

(1) The optimal packing of 17 unit squares, with side length≈4.6756
(2) The optimal packing of 11 unit squares, with side length≈3.877084

It is understandable that many have grievances with these diagrams- they subvert all expectations and lack any apparent symmetry. So what maths is done to derive these strange results?

2. Squares within Squares
2.1 Perfect Squares

For a perfect square value of *n*, we can very easily attain an optimal packing method. The minimum length is simply $\sqrt{(n)}$, providing a tidy arrangement without any surplus room.
It goes without saying that perfect squares are as easy as these problems can get.

2.2 Other Easier Solutions

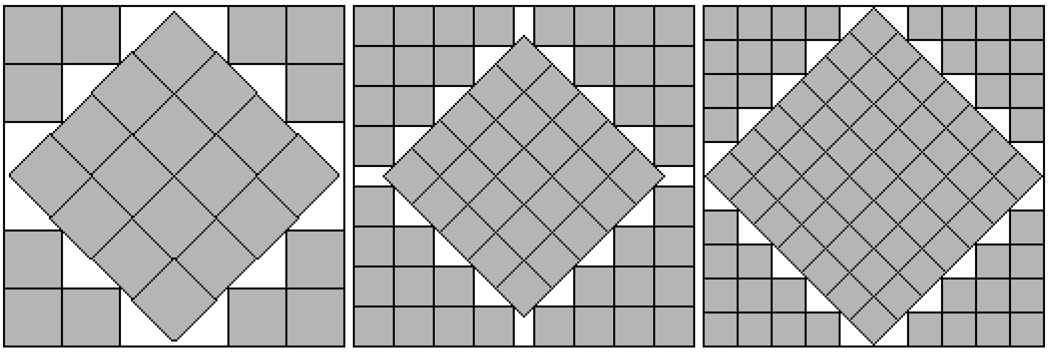
Fortunately, there are other numbers of squares which yield some fairly satisfying solutions. Frits Göbel, a Dutch mathematician who pioneered the square-packing problem and many of its early solutions, found a rather simple way to arrange squares, for certain values *n*.

For $n = 2a^2 + 2a + 2b^2$, *wherein:* $(a-1) < (b/\sqrt{2}) < (a+1)$

We can arrange the squares by creating a central *b x b* square, at a 45 degree angle, then surrounding this central square accordingly. This gives us a side length of:

$a + 1 + (b\sqrt{2})$

Not bad! This gives us a simple side length to work with, alongside a visually acceptable arrangement! For reference, some arrangements found this way are given below.



(1) The packing of n=28 squares, side length ≤ 3+2√2
(2) The packing of n=65 squares, side length ≤ 5+(5/2)√2
(3) The packing of n=89 squares, side length ≤ 5+(7/2)√2

Take note, however, that these are not optimal square packings! These arrangements provide the best <u>known</u> solutions for their values of n, not necessarily the best overall solutions. This little trick does work for *n=5*, as $2+\frac{1}{2}\sqrt{2}$ is verifiably the minimum side length, but proving this to be the case becomes far more difficult with larger values.

Whilst not perfect, these solutions allow us to construct some fairly nice inequalities which our side lengths must satisfy. Take, for example, *n=28*. We have already established that side length (L) satisfies $L \leq 3+2\sqrt{2}$. Now we need only realise that, for our lower bound, L must satisfy $L \geq \sqrt{28}$. This owes to the simple fact that a square with side length $\sqrt{n}$ can contain *n* unit squares with *no extra space (i.e the side length cannot be any smaller)*. Therefore, we establish that:

$\sqrt{28} \leq L \leq 3+2\sqrt{2}$
Or (to 2 d.p):
$5.29 \leq L \leq 5.83$

Through the $b \times b$ method, we've found some convenient upper bounds for certain n values. We can use these results to get some upper limits for other values of n. If we add an 'L' shape to these arrangements, we are able to find good packings for larger n values. For instance, adding an 'L' to the edge of the 28-square arrangement (above) finds us the best known packings for 37 squares, and 50 squares. The same logic can be applied for our other examples.

2.3 Erich Friedman

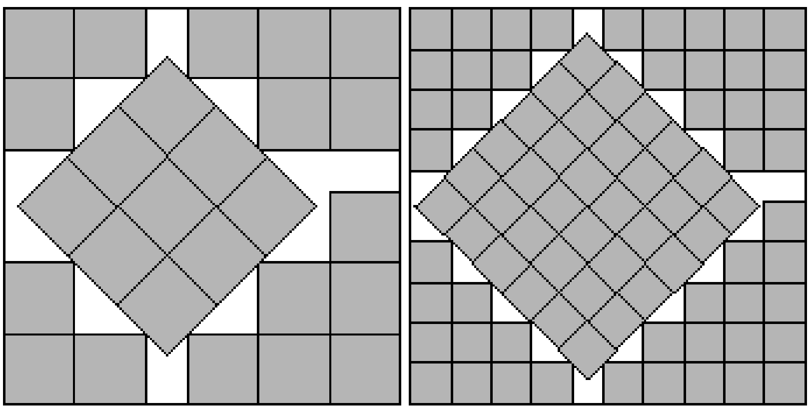
Of all the mathematicians who have worked on this problem, Erich Friedman contributed the most. Through some combinatorics wizardry, Friedman was able to find and improve upon arrangements for many values of n. He details many of these findings in his paper - "Packing Unit Squares in Squares - A Survey and new results". This paper is freely available online.

One advancement that Friedman made, was simply by adding a column to our previously mentioned $b \times b$ arrangements. Friedman realised that:

For $n = 2a^2 + 2a + b$, We can offset the $b \times b$ arrangement from the centre of the square, and fit the boxes within a rectangle with Side Lengths:

$a + \frac{1}{2} + b/\sqrt{2}$ *and* $a + 3/2 + b/\sqrt{2}$

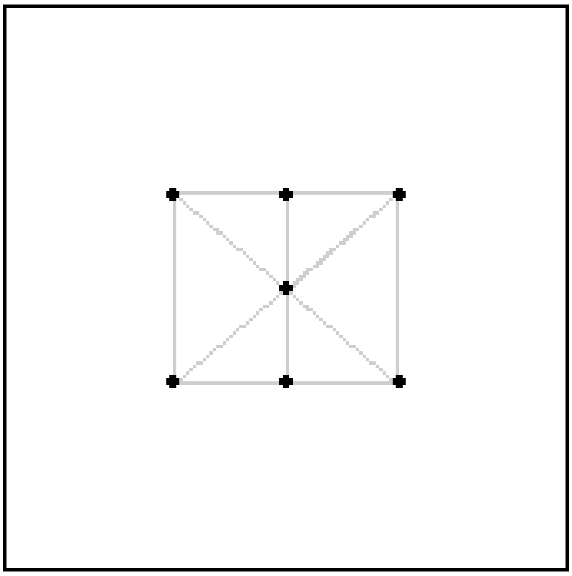
The longer of these two sides is 1 unit longer than the smaller, giving us an easy method! By simply adding a column of squares to this rectangle, we create a square! This method gives us some strong solutions for $n = 26$ and $n = 85$.



(1) The Best Known packing for 26 squares, L ≤ 7/2 + (3/2)√2
(2) The Best Known packing for 85 Squares, L ≤ 11/2 + 3√2

Friedman also made use of many other methods within combinatorics to find new solutions and improvements. He also managed to create concrete proofs for the lower bounds of certain values n which had yet not been wholly proven. This was done by establishing that certain points cannot be avoided for certain side lengths.

Take, for example, a square with side length 3. It had previously been theorised that the minimum length L for *n=8* was 3, but Friedman used unavoidable points to prove this to be the case. As shown below, there are 7 points which cannot be avoided when arranging 8 squares within a square with L=3 (Details on how these points were attained can be found in Friedman's paper). Note that these points are situated at *x-values* of 0.9, 1.5 and 2.1, and *y-values* of 1, 1.5 and 2 with respect to the fixed origin at the square's bottom-left vertex.



Through evaluating these unavoidable points, we can fairly nicely prove a concrete lower bound for our value of *n*. Friedman's method of considering unavoidable points was also used to establish Lower Bounds for *n=24, n=35, n=5, n=2, n=3, n=7, n=14 and n=11*.

Friedman used these unavoidable points, alongside other combinatorics methods, to improve bounds for numerous values of *n*. We haven't covered all of these contributions, as there were far too many, some of which having some particularly rigorous mathematical reasoning behind them.

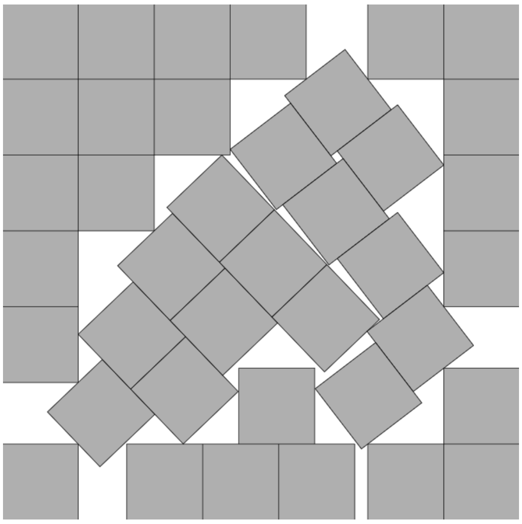
2.4 Later Developments

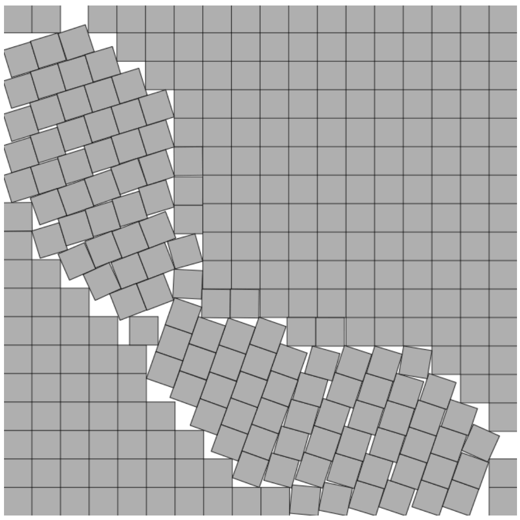
The main mathematician concerned with the recent continuation of Friedman's work has been David Ellsworth, who has optimised methods and solutions for many values, especially for those *n*>100. A major tool for those working on the problem in the 21st Century has been the use of Computer programs.

The two computer programs which are particularly relevant are the Gensane-Ryckelynck Algorithm, and Thomas Schadt's Annealing Program. At the most basic level, the programs are generally similar, with the later acting as a continuation of the former.

These algorithms, used for some value *n*, will 'start from randomness', i.e they will generate a random arrangement (which will generally be far from the best arrangement). The arrangement will then be continuously altered to improve the solution. A key aspect of

Schadt's program is its 'annealing' nature. This essentially means that the program will, on occasion, accept an arrangement which seems like a step backwards- in favour of a more optimal solution through further adjustments. This process is absolutely key in many algorithms, but in this specific context it can prevent the algorithm from getting 'stuck' at local optima, wherein there is no convenient way to improve the arrangement with simple adjustments.

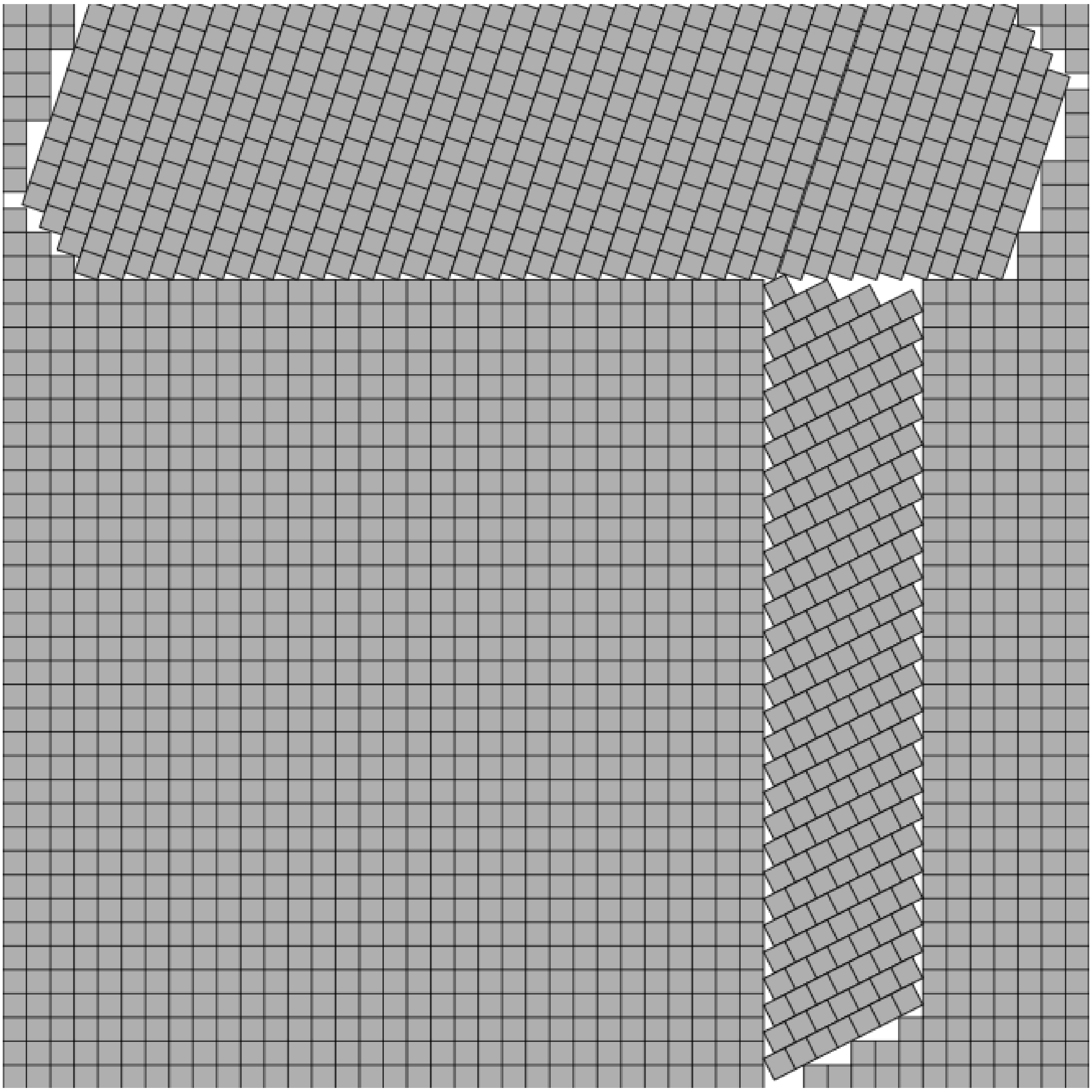




1. **The Best arrangement for 39 squares, found by Thomas Schadt using his annealing program in January 2026**
2. **The Best arrangement for 306 squares, found by David Ellsworth using a modified annealing program in February 2026**

Hopefully the two above examples can help to shine a little light on the necessity of computers in these problems. For *n=39 and n=306*, squares are rotated at multiple non-trivial angles. These results would be incredibly difficult to happen across using any traditional mathematical methods.

Nevertheless, Ellsworth was able to also come up with some incredibly impressive results without use of the annealing program. In December 2024, he found the best known solution for *n=2043*. It's worth realising that Göbel (the previously mentioned originator of the square-packing problem) had found an arrangement for *n=2043* in the Eighties (he did this using a 1-wide strip rotated at 45 degrees). Ellsworth's solution is more accurate than this arrangement.

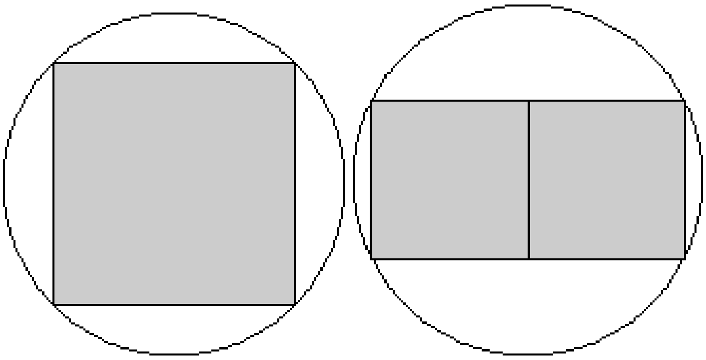


(1) This behemoth, Ellsworth's arrangement for n=2043, was worthy of a little more space than the previous arrangements have been given.

3. Squares within other shapes

Compared to the (evidently) popular discipline of squeezing squares into larger squares, the study of square-packing within other shapes is relatively neglected. For example, the study of square-packing within circles has just 2 results proven to be optimal! The only n values for which we have proven optimal radii are $n=1$ and $n=2$, each shown below. Each one has a trivial proof.

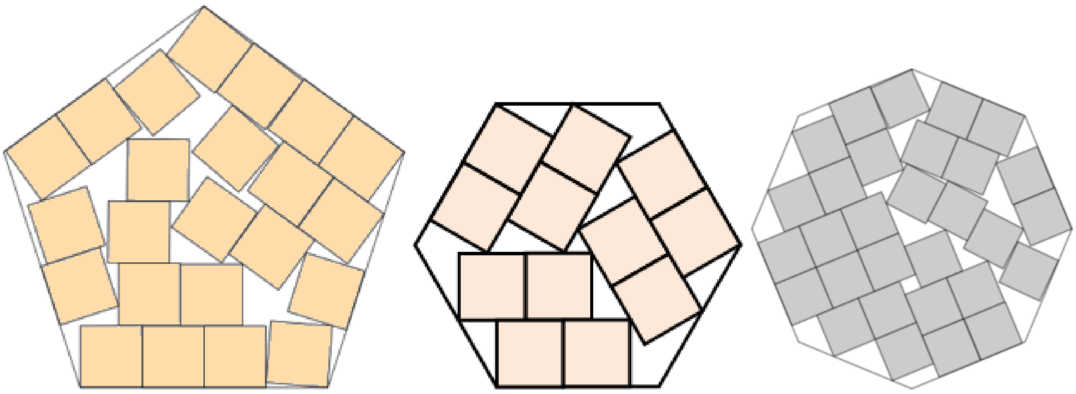
For $n=1$, we know that the square is inscribed by the circle. Thus, the circumference is equal to the diagonal of the square (which we can find to be $\sqrt{2}$ with pythagoras). Hence $r=\sqrt{2}/2$
The same logic is applied for $n=2$, giving us $r=\sqrt{5}/2$



(1) The trivial arrangement for n=1 squares in a circle
(2) The trivial arrangement for n=2 squares in a circle

Any further values *n* are without a proof (though there do exist strong current arrangements).

Other shapes that various mathematicians (primarily Erich Friedman, in truth) have laboured to insert squares into include Hexagons, Octagons, Pentagons and Tans. Collected below are some examples of the best known configurations of squares within their respective shapes. Perhaps most interesting is that each of these configurations has been found within the last 2 months (as of April 2026). This is a quickly developing discipline for those studying combinatorics!



(1) The best known configuration for n=23 squares within a pentagon, side length 4.002. Found by Ignacio Vallejo April 2026
(2) The best known configuration for n=12 squares within a hexagon, side length 2.443. Found by Maurizio Morandi March 2026
(3) The best known configuration for n=30 squares within an octagon, side length 2.74220. Found by Ignacio Vallejo March 2026.

Whilst it certainly requires a keen interest in the study of combinatorics, the arrangement of squares within other shapes is a less-developed study than that of squares within squares. In this sense, the world is your oyster (or your octagon)!

4. In Conclusion

To answer the question, 'How many squares could a square packer pack if a square packer could pack squares?', we must first answer a few other questions. Is the square packer packing their squares into another square? An octagon, perhaps? A circle? Maybe something entirely different? Furthermore, we must know how much room there is for our square packer! Does our square packer have access to Schadt's program? Whatever the answers are, we can be *fairly* certain they won't be packing many more than 2043 squares.

References

- Erich Friedman's original paper on the topic:
https://www.combinatorics.org/files/Surveys/ds7/ds7v1-1998.pdf
- Erich Friedman's Github page, for further developments and for squares in other shapes:
https://erich-friedman.github.io/packing/index.html