The Butterfly Effect

Determinism and the Limits of Prediction

Louis M. Finegan

Making predictions about the future is something we do all the time. Whether that be Quantitive Analysts using extremely complex models to predict the stock market, or meteorologists predicting if it will rain next week. Science is fundamentally based on making predictions to determine the dynamics of the universe. Predictions can be broken down into two main categories: deterministic and indeterministic systems. In this essay, I will discuss deterministic systems and discuss whether all deterministic systems actually allow us to make exact predictions about reality. In other words, if we live in a strictly deterministic universe, is the future fully well-defined provided we know the initial conditions with some level of precision?

1 Van der Pol Oscillator

In the late 1920s, a Dutch electrical engineer and physicist Balthasar van der Pol proposed the Van der Pol oscillator, while studying nonlinear circuits containing vacuum tubes. The Van der Pol oscillator is modeled by the following system of Differential Equations

$$\boxed{\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \mu(1-x^2)y - x.\end{aligned}}$$

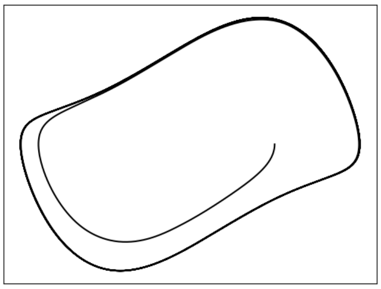

Figure 1: Van der Pol Oscillator

μ controls the strength of the nonlinearity and damping. Shown in Figure 1, the oscillator starts at a point (x, y) and is attracted to an oscillatory trajectory. This behaviour is know as a *limit cycle.* Regardless of initial starting point used, the trajectory will always fall into this oscillation. In Figure 1, I chose $\mu = 1$. However, if $\mu \approx 0$, then the system would be more like a harmonic oscillator. Despite the models applications in electrical circuits, the Van der Pol oscillator has applications in biology, such as a person's heartbeat.

We defined this behaviour of trajectories converging to a trajectory more cyclic in nature as a limit cycle. This is a form of attractor. In general, an attractor is a set of states a dynamical system, such as the Van der Pol Oscillator, naturally evolves to over time. We will dicuss another famous attractor that was discovered while predicting the weather: the Lorenz Attractor.

2 Predicting the Weather

Predicting whether it will rain next week, seems, at first, like a coin toss. There are multiple outcomes, for each of the different weather patterns. It could be raining today but tomorrow, it could be sunny, overcast, stormy, etc. However, if we apply the laws of physics, we realise that we can characterise these different weather patterns by measuring the atmospheric temperature T, the pressure p and wind speeds $\mathbf{u}$, to name a few. We can construct a deterministic system to predict the weather. Such model is the famous Navier-Stokes equation

$$\boxed{\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + g\alpha T\hat{e}_{\mathbf{z}}, \\ \nabla\cdot\mathbf{u} &= 0, \\ \frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T &= \kappa\nabla^2 T. \end{aligned}}$$

These incredibly complicated equations are one of the Millennium Problems [1]. If we measured the initial temperature, pressure, wind speed and so on, we can apply them to this system of differential equations, under a certain number of assumptions, we can predict the weather in the future.

3 Lorenz System

Figure 2: Edward Lorenz

In the early 1960s, a mathematician turned meteorologist, Edward Lorenz simplified these fluid equations[2] and created a weather simulation he could run on a computer. The simplified system of differential equations are written in terms of parameters $X(t)$, $Y(t)$ and $Z(t)$, with depend on time t. The Lorenz systems takes the form

$$\boxed{\begin{aligned} \frac{dX}{dt} &= \sigma(Y-X), \\ \frac{dY}{dt} &= X(\rho - Z) - Y, \\ \frac{dZ}{dt} &= XY - \beta Z. \end{aligned}}$$

The coefficients σ, ρ and β relate to physical properties of the fluid. Originally Lorenz used the following values for these parameters, $\sigma = 10$, $\rho = 28$ and $\beta = \frac{8}{3}$.

[1] Seven difficult unsolved problems in mathematics announced by the Clay Mathematics Institute (CMI) in 2000. For those who solve them, there is **$1 Million** prize and a fields medal, which is the Nobel Prize for Mathematicians.

[2] The atmosphere exhibits physical dynamics similar to those of fluids.

Lorenz set up his simulation and allowed it to run as he went off to make coffee, using a particular set of initial conditions. When he later restarted the computation to examine part of the output in more detail, he entered initial conditions rounded to fewer decimal places. He expected the error of the two runs to be neglible. However, that was not the case. After a short while the two solutions diverge from eachother. This unexpected sensitivity revealed that tiny changes in the initial conditions can effect the predicted outcome later in time. It is for this reason, we cannot reliably predict the weather for more than seven days.

Shown in Figure 3, A trajectory of the Lorenz system with initial conditions $X(0) = 1$, $Y(0) = 1$ and $Z(0) = 1$ in blue, and another with sligthly perturbed initial condition $X(0) = 1.056$ in orange. As seen in the regions where the curves do not overlap, the trajectories diverge from one another. Despite this divergence, both trajectories are confined to this "butterfly shape". Similiar to the limit cycle, the Lorenz system is a Attractor, but since there is no set all trajectories converge to, this is called a strange attractor, or in this case, the Lorenz Attractor.

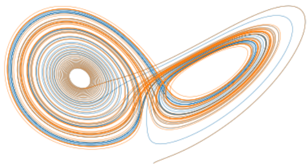

Figure 3: Lorenz Attractor Blue with perturbed initial conditions Orange.

In fact, the butterfly shape of this strange attractor is what lead to this behaviour of sensitivity of initial conditions to be coined "The Butterfly Effect".

4 The Butterfly Effect

Figure 4: A Sound of Thunder by Ray Bradbury.

At the 139th American Association for the Advancement of Science in 1972, Lorenz gave a talk titled, *Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?* The phrase refers to the effect of a butterfly's wings create tiny changes in the atmosphere that may alter the weathers state somewhere else in the world. This idea has a profound effect on popular culture. There are countless fictional sources discussing the theme of small changes having a big effect in the future.

One such example is the science fiction short story *A Sound of Thunder*, by American writer Ray Bradbury. In the year 2055, time travel has became practical, and a company called "Time Safari Inc." lets wealthy hunters go back to pre-historic times to hunt dinosaurs. A hunter names Eckels and a group travel back in time to hunt a Tyrannosaurus rex. Hunters are advised to stay on the levitating path and only shoot the animals destined to die naturally. During the hunt, Eckels panics and returns to the time machine, while the group kills the T-rex. Eckels had stepped off the path into the dirt. When the group returns to 2055 after killing the dinosaur, they realise a severly different reality. Eckels looks to his boot to discover, a dead butterfly. This subtle change caused serious alterations to their timeline.

In philosophy, James Clerk Maxwell [3] argued in his essay *Does the Progress of Physical Science Tend to Give Any Advantage to the Opinion of Necessity over that of the Freedom of the Will?* that even if natural laws are strictly deterministic, predictability is limited by the instability of certain systems and by the impossibility of knowing initial conditions with absolute precision. Maxwell distinguished between stable and unstable systems. In stable systems, small variations in initial conditions produce only small differences in the system's future behaviour. In contrast, unstable systems amplify small differences in initial conditions, leading to significantly different outcomes over time.

Figure 5: James Clerk Maxwell

A useful analogy is that of a ball on a hill. If the ball is balanced at the top of the hill, a tiny perturbation in its position will cause it to roll down one side or the other, dramatically altering its future state. This represents an unstable system. By contrast, a ball resting in a valley behaves differently: small perturbations simply cause the ball to move slightly before settling back near its original position. This represents a stable system, where small changes in the initial state lead only to small changes in the future behaviour of the system.

5 Conclusion

Sensitivity of initial conditions is not unique to the Lorenz System. Henri Poincaré first encountered similiar behaviour while studying the orbits of three celestial bodys. He discovered that tiny differences in the initial positions of bodies could lead to vastly different orbital trajectories over time. This problem is called the three body problem. This phenomena apprears in fluid motion. Small disturbances in a fluid can grow and lead to irregular flow known as turbulence. Although the equations governing these systems are deterministics, their sensitivity makes long-term prediction practically impossible, since the initial state can never be measured with infinite precision. The study of such deterministic yet (in the long term) systems is known as chaos theory.

[3] Yes. The same Maxwell who formulated Maxwell's equations in electromagnetism.

References

Bradbury, Ray (1952). *A Sound of Thunder*. First published in Collier's magazine, June 28, 1952. Collier's.

Campbell, Lewis et al. (2010). *The Life of James Clerk Maxwell: With a Selection from his Correspondence and Occasional Writings and a Sketch of his Contributions to Science*. Cambridge Library Collection - Physical Sciences. Cambridge University Press.

Chaos theory (2019). Wikipedia. URL: https://en.wikipedia.org/wiki/Chaos_Theory.

Circa January 1961: Lorenz and the Butterfly Effect (2024). Aps.org. URL: https://www.aps.org/archives/publications/apsnews/200301/history.cfm.

Edward Norton Lorenz (2021). Wikipedia. URL: https://en.wikipedia.org/wiki/Edward_Norton_Lorenz.

Park, David (2025). *Analysis of the Van der Pol Oscillator*. DOI: 10.1149/osf.io/dra3h_v1.