The 4 Colours of a Proof

Introduction

Imagine colouring in a picture, one that is split into several sections, in which you fill each region with a solid colour. On the one hand, you might try to achieve a sense of realism, maybe creating a simplified version of the original picture, or maybe, you wonder just how many colours you would need to fill in the picture in a way that no connected areas share the same colour.

In 1852, when colouring a map of English counties, Francis Guthrie stumbled upon the Four Colour Problem, a problem that would remain unsolved for over a hundred years. Upon discovery, Guthrie wrote a letter to his brother, a student at University College London, for further investigation. This would then lead to an extensive string of failed proofs, spanning more than a century, before it was finally solved.

But what is the infamous Four Colour Problem that stumped mathematicians for years? To put it simply, the Four Colour Theorem stated that any planar graph could be coloured with at most four different colours, in a way that no two adjacent vertices would have the same colour. In the original problem, the planar graph was represented as a map (figure 1), with the vertices representing regions and the edges representing the shared boundaries, – contact at points isn't considered. As a concept, it isn't difficult to grasp, however, proving that it holds true for every possible map took over a century.

Over 120 years after it was first conjectured, the Four Colour Problem was finally solved, but this time, it was different. The Four Colour Theorem was the first major proof to be significantly assisted by a computer. This meant that a large number of mathematicians had initially rejected the idea, asking, if it could not be proven by a human, was it truly a proof?

Much like the original problem, I will explain the four "colours" that make up this proof.

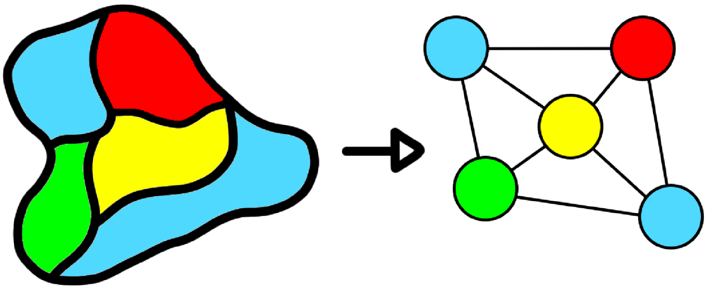

Figure 1: how a map can be transformed to form a graph

The First Colour – Logic

Over the course of many years, mathematicians reformulated the original problem posed by Guthrie as a graph theory problem, with the regions represented as vertices, and the boundaries represented as edges. But how is this simplification equivalent to the complexity of a map? A vertex simply represents an item, in this case, a region on a map and an edge shows the connection between multiple items, or multiple regions. Since boundaries on a map must be a physical connection that cannot cross, the resulting graph must be planar.

But how would anyone even begin to prove this? In a formal, mathematical sense, a proof is defined as a chain of deductive reasoning, in which each step is adhering to strict rules and clearly continued from the previous. A proof begins from axioms, a fundamental basis that is taken to be true, and proceeds using further axioms, assumptions or derivations. This meant that while just about anyone could test and see that the Four Colour Theorem would be true for any given map, it is exceedingly difficult to prove that it remains true for every single possible map.

Some might think that with enough trial and error, we could deduce that this would be the case for all maps. However, there is effectively an infinite number of various maps that anyone could conjure up, meaning that checking every possible map would be quite literally, impossible. This is where Heinrich Heesch later showed that all planar graphs could be reduced to an unavoidable set, thereby reducing the number of possibilities, from an infinite amount to roughly 9000 configurations, finally making a full proof conceivable.

The Second Colour - Structure

To prove the Four Colour Theorem, one must first understand the Five Colour Theorem. In 1890, Percy John Heawood utilises mathematical induction to prove the Five Colour Theorem. The first step? He manages to prove that for any graph with just one vertex, there is an arrangement in which the vertex can be filled with a single colour, using less than five colours overall - shown in figure 2. Next, he assumes that for any graph with *n* vertices, there is a colouring that satisfies the Five Colour Theorem, a crucial component in proving by induction.

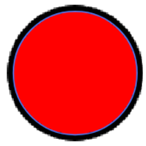

Figure 2 : base case – a single vertex

He then applies this same logic to a graph with *n+1* vertices. A graph of *n+1* vertices can be split into a graph of *n* vertices, which can be coloured such that it only requires five colours, and a single vertex. In every planar graph, there will always be a vertex with a degree of five or less, as shown in figure 3, this will be the vertex that we separate from the graph. If this vertex has a degree less than five, it can simply be filled with the final colour that it isn't adjacent to. However, if it has five adjacent vertices, it utilises a technique called a Kempe chain. To explain a Kempe chain in short, imagine that all five vertices that are connected to the isolated one are arranged in a circular pattern, now look at the colours of the first and third vertices, let's say that they are red and blue. Following this, consider all the vertices that are red and blue in the entire graph, and find the longest possible path using only edges connecting vertices of those two alternating colours. If the colour of the third vertex, blue in this case, isn't in the chain, swap the colours and now the entire graph can be filled using five colours. If that

Proving that every planar graph contains a vertex with degree 5 or less

Let n be the number of vertices and m the number of edges of a planar graph G

We will prove this by contradiction. Let's say that every vertex of G has degree at least 6

$$\delta(G) \geq 6$$

The Handshake Lemma states that the the sum of the degrees of all the vertices is equal to twice the number of edges, so

$$\sum \deg(v) = 2m$$

Since each vertex needs to have a degree of at least 6, this means that

$$2m = \sum \deg(v) \geq 6n$$

$$m \geq 3n$$

However, for any planar graph with $n \geq 3$, Euler's formula states that

$$m \leq 3n - 6$$

and as every vertex has a degree of at least 6, this means that $n \geq 7$, and so this must be true

These two inequalities give us

$$3n \leq m \leq 3n - 6$$

which is impossible.

This contradiction shows that our original assumption was false and therefore, we have proven that every planar graph must contain at least one vertex of degree 5 or less

Figure 3: proving that all planar graphs must contain a vertex with degree 5 or less

colour is in the chain, repeat the steps for the second and fourth vertices. At least one of these pairs must be valid, as if neither were, the Kempe chains would cross, which is impossible for a planar graph.

So why does a similar logic not apply to the Four Colour Theorem? Proving that the Five Colour Theorem is considerably more elegant and sophisticated as it relies on the fact that one vertex will have a degree of five or less. On the other hand, a Four Colour Theorem wouldn't be able to apply the same idea as planarity only guarantees that one of the vertices has a degree of five or less, but not necessarily four.

This then led to the idea that if the Four Colour Theorem is false, there must be at least one map with the least number of regions that requires at least five colours. How would they find this map? By analysing every single possibility. This map would have to contain at least one configuration from the unavoidable set; these were specific arrangements that every planar graph always includes. However, this ideal map would also have to omit all reducible configurations. If the map contained one, it could be reduced to a smaller graph that also required more than four colours, contradicting the statement that the original map was the smallest.

In order to prove this, it was theorised that someone would have to tediously check through thousands of graphs, a task that would take months, or even years, of painstaking manual labour. With this in mind, the next colour of proof will address the role of computation within the proof, and why computer assistance became an essential part of it.

The Third Colour - Computation

1976, Kenneth Appel and Wolfgang Haken announced to the world that they had solved the infamous Four Colour Problem. However, the initial response wasn't what they had anticipated. Many mathematicians had at first, rejected their proof, and others refused to report on it, in fear of the proof being flawed like many before it.

Appel and Haken assumed that a minimal counterexample to the theorem must exist, the smallest map that would need more than four colours to completely fill in. They proved that the smallest planar graph that cannot be coloured with four colours would have to contain at least one of 1482 (originally 1936) possible configurations, that form what is known as the unavoidable set – all planar graphs must contain at least one of the configurations in this set. Following this, they managed to prove that all 1482 of these possible configurations would be reducible, however, we already stated that our original counterexample would be the smallest, and therefore couldn't be reduced. This contradiction proved that the smallest map that contradicts the Four Colour Theorem couldn't exist, therefore proving the theorem.

But why was this initially rejected? The proof itself is logically sound, however, individually testing thousands of complex graphs for reducibility is a task far too tedious to complete by hand. They therefore turned to a computer program to analyse each and every configuration that stated whether or not it was reducible, and after 1000 hours of computer time, the program confirmed that all 1482 graphs were in fact reducible, finally proving the Four Colour Theorem. Mathematicians disliked the notion that the proof wasn't verifiable by a human, and that they would have to trust the code would verify it for them. In its entirety, the proof wouldn't be readable by a single human, even over a lifetime, and this created a sense of unease within the mathematical community; if a proof wasn't understandable by any one person, was it really proving anything? It's important to note as well that future refinements reduced the unavoidable sets and made it easier to verify.

The Fourth Colour - Understanding

So, why does this prove the Four Colour Theorem? Appel and Haken proved that no smallest counterexample can exist, meaning that there is no map that cannot be coloured with only four colours. However, this method of verification harshly contrasts the proof for the Five Colour Theorem, which was a short, elegant solution. It raises the question as to whether mechanical verification is on par with human reasoning; is a mass scale verification of every possibility the same as truly understanding why it works? On the other hand, why should it matter? Whether we understand why the proof works or not, it still proves the Four Colour Theorem, so why should the elegance of a solution matter more than the content?

Conclusion

Throughout this essay, I hope that I have managed to explain the notorious Four Colour Problem in a simple manner, such that almost anyone could comprehend the maths behind it, and recognise the sheer brilliance that was necessary to solve it. From a different perspective, the Four Colour Problem isn't a difficult concept to grasp, it's an idea that almost anyone could understand, but it goes to show how even the simplest of conjectures can stump mathematicians for years to come. The Four Colour Theorem wasn't merely a basic colouring problem; it wasn't just a milestone in graph theory either, it redefined the idea of proof itself, forcing mathematicians to confront the idea of what it truly means to prove something.