Why $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

By Tej Pandey

1. The problem

In 1650, Italian mathematician Pietro Mengoli posed the question - 'What is the sum of all the reciprocals of square numbers?' Later on, this problem came to be known as the Basel problem, a name which had no particular backstory behind it, but was decided upon because many of Megoli's contemporaries, including the Bernoullis and Euler, who worked on the problem, were themselves from the city of Basel.

But what makes this problem especially fascinating is how it was solved (many times over actually). The first solution, found by Euler in 1734, which I will explore in this essay, combines calculus, trigonometry, and some of his own genius. However, there are many other solutions that arise from completely different fields of mathematics.

Before delving into Euler's own proof, we will have to take a bit of a tangent.

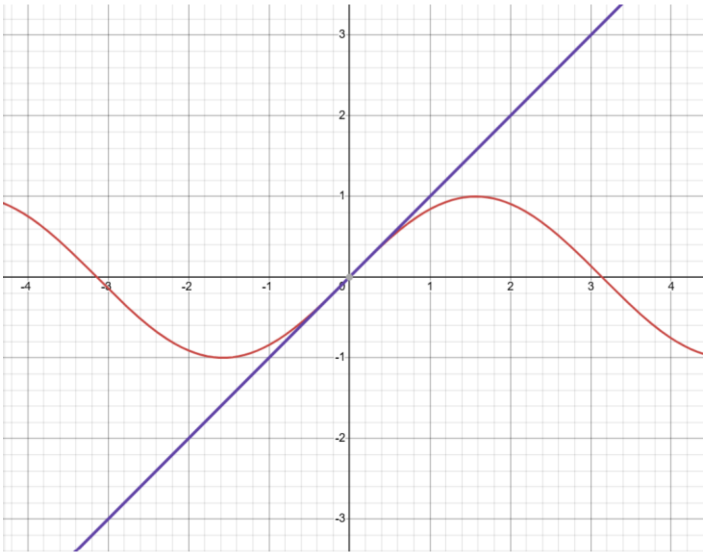
2. Taylor series – writing trigonometric functions as polynomials using calculus

How do you think the sin function on your calculator works? I know it seems completely unrelated to the Basel problem, but stay with me for a bit.

One idea is that it could have an inbuilt 'sin table' –of the value of sin x for every angle x (this is what my dad claimed people used in school before calculators came out). For example, you could have sin 60= (√3)/2, and sin 30 = 1/2, and maybe even one sin value for every angle with an integer value. But there is a limit to how much you can store on a calculator. And I can input as many decimal places as I want into my sin function: my calculator just doesn't have enough space for things like sin(65.394832) = 0.909198557. So how does it do it?

What about a function that very closely, maybe even perfectly approximates sin x, allowing a calculator to use run-of-the-mill arithmetic to find a value of sin x for any x? One of simplest types of functions are polynomials – expressions which only include variables, the 4 operations of arithmetic, and positive integer exponentiation. For example, , these are all polynomials. So, let's try to make an approximation of the sin function using a polynomial.

A good place to start is the y-intercept of sin x: we know that sin(0) = 0, therefore, the y-intercept is 0 and our polynomial has a constant term of 0. The simplest polynomial with a constant term of 0 is just y = x. Remarkably, this is a pretty good guess for small values of x:



y=sinx(red) against y=x(blue)

But we can certainly improve our polynomial approximation. And to do this, we need calculus, and an ingenious trick.

Here's the idea: if we suppose that our polynomial $P(x) = ax + bx^2 + cx^3$ ... ,(remember, it must have no constant term), actually exists and is equal to sin(x) for all x, then its n-th derivatives at x = 0 would be the same as those of sin x at x = 0. And this can show why our guess actually worked.

If you recall polynomial differentiation: $\frac{dy}{dx}x^n = nx^{n-1}$ we can find that the first derivative $P'(x) = a + 2bx + 3cx^2$ ... is in fact simply equal to a when x = 0 (all the other terms with a positive x exponent are simply just 0). Now, since the first derivative of sin x = cos x, and cos 0 = 1, we know that a = 1, so the polynomial starts with x (which was our lucky guess). Next, we can do this for higher order derivatives. Now, the orders derivatives of $\sin x$ are as follows:

Order of derivative	1	2	3	4
Function	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
Value at 0	1	0	-1	0

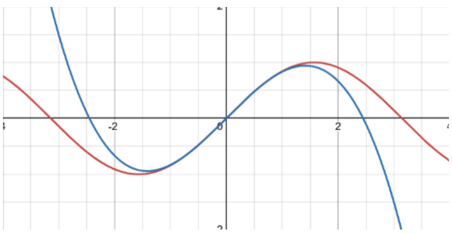
And they simply repeat after this, forming a periodic sequence with period 4. Now, we can use this to infer the following terms of our polynomial. Let's start with the $n'th$ derivative of our polynomial at 0:

$$\frac{d^n y}{dx^n}P(x) = an! + b\frac{(n+1)!}{1!}x + c\frac{(n+2)!}{2!}x^2 \ldots,$$

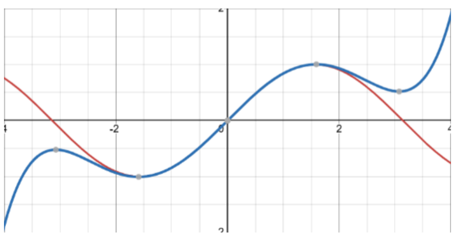
where a, b, c ... are the coefficients of x^n, x^{n+1}, x^{n+2} ... respectively. Now, since x = 0, this sum simply becomes $an!$. Due to the periodicity of derivatives of sin, whenever n is even, $\frac{d^n y}{dx^n}P(x) = 0$, so the coefficient of even exponents in the polynomial is 0. However, when n is one more than a multiple of 4, $an!$ = 1 so $a = \frac{1}{n!}$, and when n is three more than a multiple of 4, $an! = -1$ so $a = -\frac{1}{n!}$. However, you may wonder how many terms our polynomial have? The answer is infinitely many, and each successive term in the series improves accuracy. Therefore, with enough terms, a calculator can do an extremely good approximation of $\sin x$ using just basic arithmetic. How do we notate such a sum? Like this:

$$\sum_{n=0}^{\infty}\frac{-1^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots .$$

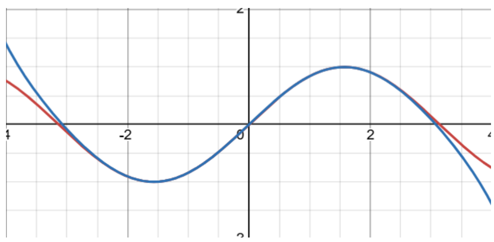
On Desmos, you can see how each term adds more and more accuracy:



$y = x - \frac{x^3}{3!}$



$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

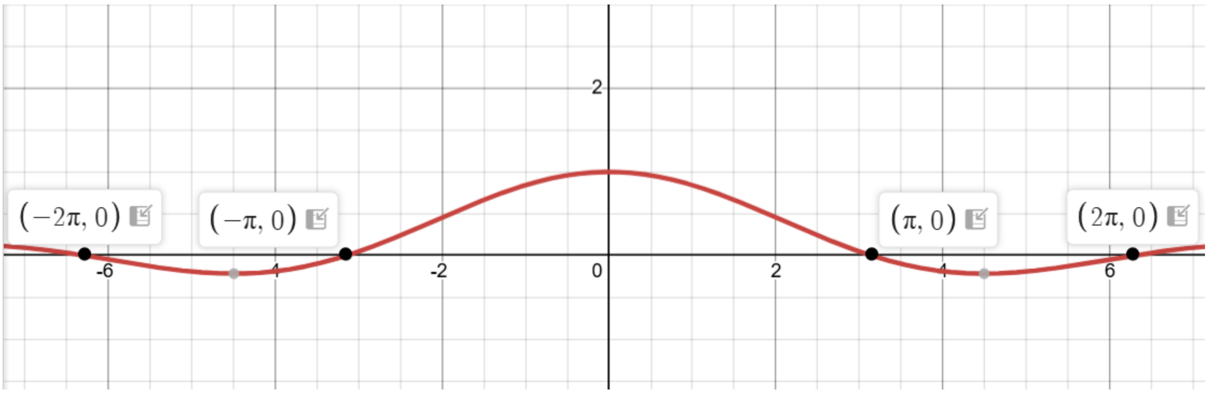


$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7}$

While this may be a cool result, how does it actually help us in solving our original problem? That's where Euler's trick comes in.

3. Euler's trick

Here's the graph of $\frac{\sin x}{x}$:



As you can see, the x-intercepts of $\frac{\sin x}{x}$ always lie on integer multiples of π, because sin $n\pi = 0$ whenever n is an integer. Now, Euler's idea was to use this to treat $\frac{\sin x}{x}$ as a infinite polynomial product with roots at integer multiples of pi, therefore:

$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right) \ldots = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{\pi^2}\right) \ldots,$$

which in product notation becomes:

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Now, this may seem like a leap, an in fact Euler himself had no substantial proof for it other than the fact that both the infinite product and $\frac{\sin x}{x}$ had the same roots, and that they were extremely similar-looking functions. Later on, formal proofs of this 'trick' were found using deeper analysis.

Getting back on track, we have just one step left to connect Euler's result to the Basel problem.

If we expand our product:

$$\prod_{n=1}^{\infty}(1-\frac{x^2}{n^2\pi^2}),$$

just enough, we can find the coefficient of x^2. But how do you multiply an infinite product? We can do this by imagining that we 'choose' one term out of $-\frac{x^2}{n^2\pi^2}$, 1, from every bracket, and sum up all our possible choices. To get a choice that has x^2 multiplied by a coefficient only, we must choose one$-\frac{x^2}{n^2\pi^2}$ for a certain n in one of the brackets, and choose 1's from all the other brackets. If we do this for every choice of n, we find that our overall x^2 coefficient is:

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2},$$

which is:

$$\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

so the very Basel sum we want, multiplied by $\frac{1}{\pi^2}$. We can furthermore relate this to the Taylor series expansion of $\frac{\sin x}{x}$, since

$$\frac{\sin x}{x}=\frac{1}{x}\sum_{n=0}^{\infty}\frac{-1^n x^{2n+1}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\dots.$$

As you can see, the x^2 coefficient here is$\frac{1}{3!}$ so $\frac{1}{6}$. Labelling the Basel sum as B,

$$\frac{1}{\pi^2}B=\frac{1}{6},$$

$$B=\frac{\pi^2}{6}.$$

This is the exact result we wanted, and for Euler, this was one of his first major breakthroughs into the halls of mathematical glory.

4. Conclusion

The Basel problem is memorable for its result – which is a striking unity between the reciprocals of integer square numbers (a problem seemingly confined to the realms of number theory and algebra), and the geometric constant of π. However, it is also the problem that shot Euler to immediate fame, and although his argument was later placed on firmer rigorous foundations, his work on this problem still proved influential a century later when Reimann was studying his famous (or infamous) zeta function – but that's a whole other, far deeper story.