

Infinity Isn't One Thing: The Strange World of Bigger and Smaller Infinities

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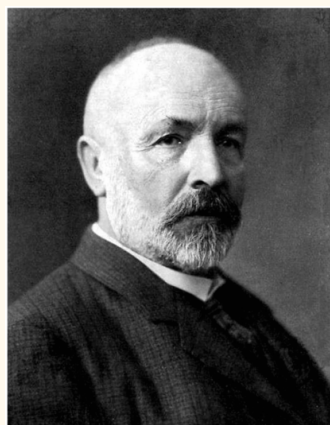


When I first heard about infinity as a child, I thought it was simple—just a really, really big number that never ended. Like the stars in the sky or the pages in an endless book. But everything changed the day I learned about Hilbert's Infinite Hotel.

Imagine a hotel with infinitely many rooms, all completely full. Every room has a guest. Now, a new guest arrives at the front desk. Instead of turning them away, the manager simply asks each current guest to move to the next room: the person in room 1 moves to room 2, room 2 to room 3, and so on, forever. Suddenly, room 1 is free, and the new guest can check in with ease.

This sounds impossible — almost like a magic trick. But it's real mathematics. And it's only the beginning. To understand why, we need to rethink what infinity really means.

What Do We Actually Mean by “Infinity”?



For most of my life, I believed infinity was just one thing—an endless stretch that you could never reach. But when I started reading about Georg Cantor, I realized how wrong I was.

Cantor was a German mathematician in the late 19th century who dared to treat infinity as a serious mathematical object. At the time, many mathematicians feared infinity because it kept leading to paradoxes. But Cantor asked a radical question: What if we can compare different infinities the same way we compare finite numbers?

Welcome to Hilbert’s Grand Hotel

Let’s return to the hotel. It has infinitely many rooms, numbered 1, 2, 3, 4, ... stretching on forever, and every room is occupied.

Now imagine not one new guest, but an infinite bus full of new guests arriving. How can the hotel possibly accommodate them?

The manager has a clever solution: he asks every current guest in room n to move to room $2n$. So the guest in room 1 goes to room 2, room 2 to room 4, room 3 to room 6, and so on. All the odd-numbered rooms — 1, 3, 5, 7, ... — become completely empty. And because there are infinitely many odd numbers, the entire infinite group of new guests can now check in comfortably.

This simple thought experiment breaks our everyday understanding of “full.” Even when the hotel is completely occupied by infinitely many guests, it can still welcome infinitely many more. Our intuition about space and limits completely collapses.

But we can go even further — let’s call it **Hilbert’s Hotel Level 2**. What if infinitely many buses arrive, and each bus itself carries infinitely many passengers? Now we have infinity multiplied by infinity.

At first, this seems impossible to organize. But mathematics has a beautiful trick. We can pair each new guest with a unique room using two numbers: the bus number and the seat number. By arranging them cleverly (for example, moving along diagonals in

an infinite grid), every single person from every bus gets a room. This shows something astonishing: even **infinity** $\times$ **infinity** is still the same size as the original countable infinity. It remains $\aleph_0$.

Not All Infinities Are Created Equal

So far, we've been dealing with **countable infinity** — the size of the natural numbers (1, 2, 3, ...), which mathematicians denote as $\aleph_0$ (aleph-null).

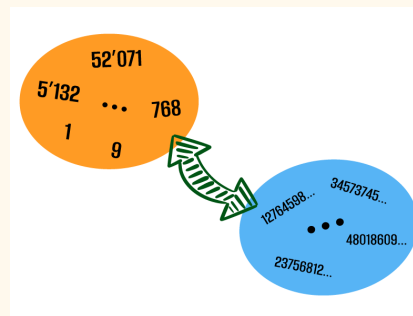
But here comes the moment that still gives me chills.

Cantor proved that there exists a larger infinity: the infinity of the real numbers between 0 and 1. These include every possible decimal—0.1, 0.25, 0.333..., $\sqrt{2}-1$, and infinitely many more that we can never list completely.

To show this, he used one of the most elegant arguments in mathematics: **Cantor's diagonal argument**.

Suppose someone claims they have made a complete list of all real numbers between 0 and 1. It might look something like this:

- 0.1234567...
- 0.5555555...
- 0.1010101...
- 0.3141592...



Cantor suggests: let's build a new number that is **not** on this list. We change the first digit of the first number, the second digit of the second number, the third digit of the third number, and so on, moving diagonally. The new number we create will differ from every single number on the list in at least one decimal place.

No matter how complete the list seems, we can always find a number that is missing.

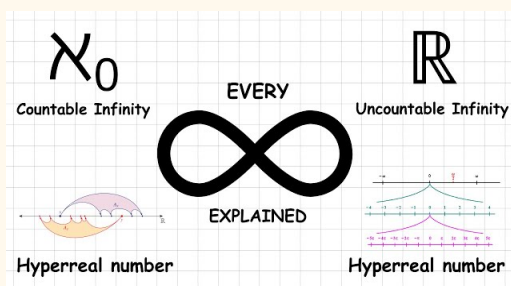
This is the moment where mathematics does something shocking: it proves that even infinity can be outgrown. No matter how hard you try to list every real number, infinity will always escape your attempt to contain it. There are simply more real numbers between 0 and 1 than there are natural numbers in the whole infinite sequence.

In other words, some infinities are bigger than others.

Many mathematicians rejected Cantor's ideas at first, believing they were too strange and paradoxical to be true. Yet today, his work forms the foundation of modern set theory.

Why Does This Matter?

Discovering that infinities come in different sizes didn't just change mathematics — it changed how I see the world.

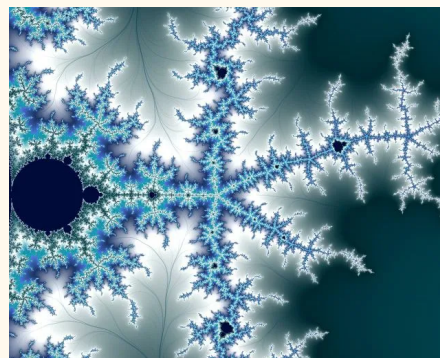


Our brains evolved to handle finite things: counting money, people, or days. When we meet true infinity, our intuition fails dramatically. What feels obvious—that “infinity is infinity”—turns out to be false. Our intuition is built for survival in a finite world, not for understanding something that breaks every linear rule we know. This teaches us humility. It

reminds us that there are truths in the universe that lie far beyond our everyday thinking.

In computer science, infinity appears in algorithms that run forever or in recursive functions that keep calling themselves. In nature, we see hints of it in fractals—patterns like coastlines or snowflakes where zooming in forever reveals the same complexity again and again. These real-world connections show that infinity is not just abstract theory; it quietly shapes the digital and physical world around us.

Cantor's work also opened doors to some of the deepest questions in mathematics, such as the continuum hypothesis: is there any infinity between $\aleph_0$ and the larger continuum? Questions like these pushed mathematics into new territories and even influenced modern logic and computer science.



But above all, this story shows the beauty of mathematics. It is not just about calculations—it is about daring to explore ideas that seem impossible and finding wonder in the process.

And yet, despite all this, we are still only beginning to understand infinity.

Conclusion

Infinity isn't one thing. It comes in different sizes, different strengths, and different levels of strangeness. From the playful countable infinity of Hilbert's ever-expanding hotel to the vast, unlistable ocean of real numbers, infinity reveals a rich landscape that continues to surprise us.

The next time someone says "infinite," I pause and smile. Are they thinking of the smaller infinity of counting numbers, or something far greater? Perhaps it reveals the boundaries of how we understand reality itself.

And that realization is why infinity will always be bigger than anything we can ever imagine.